THE BIAS AND STABILITY OF JACK-KNIFE VARIANCE ESTIMATOR IN RATIO ESTIMATION

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Summary

The Jack-Knife variance estimator $v_2(r)$ (say) based on splitting the sample at random into g groups is applied to estimation of the variance of the ratio estimator $r=\bar{y}/\bar{x}$ of the population ratio $R=\bar{Y}/\bar{X}$. Assuming a linear regression of y on x where x has a gamma distribution it is shown that the exact bias of $v_2(r)$ is a decreasing function of g. The exact bias of $v_2(r)$ with g = n is less than that of $v_1(r)$, the customary variance estimator of r, for moderate sample sizes. The exact stability of $v_2(r)$ for the special case of g = 2 is shown to be less than that of $v_1(r)$. The asymptotic stability of $v_1(r)$ is also discussed for a bivariate normal distribution.

1. Introduction

Ratio estimators are often employed in sample surveys for estimating the population mean $\bar{Y}$ of a characteristic of interest 'y' or the population ratio $R=\bar{Y}/\bar{X}$ utilizing an auxiliary variate 'x' that is positively correlated with 'y'. The estimate of the variance associated with an estimator is often used in drawing statistical inferences (e.g. confidence limits on the estimand). It is, therefore, desirable that a variance estimator should be as stable as possible. In this paper we investigate the bias and stability of the Jack-Knife variance estimator and the customary variance estimator in ratio estimation.

We shall confine ourselves to simple random sampling and assume that the population size N is infinite to simplify the discussion. From a simple random sample of n pairs (y_i, x_i) we have the customary ratio estimator of R as

$$r = \bar{y}/\bar{x} \tag{1}$$

where $\bar{y}$ and $\bar{x}$ are the sample means of y and x respectively. As an estimator of V(r), the variance of r, it is customary to take

$$v_1(r) = (s_y^2 - 2rs_{yx} + r^2 s_x^2)/(n\bar{X}^2) \tag{2}$$

where s_y^2 and s_x^2 are sample mean squares and s_{yx} is the sample covariance. It is known that the bias of $v_1(r)$ is of order 1/n.

Let the sample of size n be divided at random into g groups, each of size p so that n=pg. Let

$$r_Q = g^{-1} \sum_1^g r_{Qj} \tag{3}$$

where

$$r_{Qj} = gr - (g-1)r_j' \tag{4}$$

and r_j' is the customary ratio estimator calculated from the sample after omitting the jth group.

Tukey (1958) has pointed out that the estimators like r_{Qj} (called pseduo - values) may, to a good approximation, be treated as though they are g independent estimators. Therefore, we can use the simple estimator

$$v_2(r) = g^{-1}(g-1)^{-1} \sum_1^g (r_{Qj} - r_Q)^2 \tag{5}$$

as an estimator of V(r) since $r_Q = \sum_1^g r_{Qj}/g$. Tukey has called this general procedure, described here in the context of ratio estimation, the 'Jack-Knife'.

The stability of a variance estimator may be judged by its coefficient of variation. Kokan (1963) has investigated the large-sample stabilities of $v_1(r)$ and the unbiased variance estimator:

$$v(\bar{y}/\bar{X}) = s_y^2/(n\bar{X}^2) \tag{6}$$

where $\bar{y}/\bar{X}$ is the ratio estimator of R not using the sample x-information. He has shown that the coefficient of variation of $v_1(r)$ is <u>always</u> larger than that of $v(\bar{y}/\bar{X})$ for a <u>bivariate</u> normal distribution and this property also holds for a bivariate log normal distribution for certain ranges of the parameters. Lauh and Williams (1963) have made a Monte Carlo study of the stabilities of $v_1(r)$ and $v_2(r)$ with g=n for small samples. Assuming that the regression of y on x is through the origin and C_x is small, they have shown that the Monte Carlo variances of $v_1(r)$ and $v_2(r)$ are about the same when x has a normal distribution whereas the variance of $v_2(r)$ is considerably smaller than that of $v_1(r)$ when x has an exponential distribution.

Recently Rao and Beegle (1966) investigated the small sample properties of $v(\bar{y}/\bar{X})$, $v_1(r)$ and $v_2(r)$ by a Monte Carlo study. They have shown that under the Lauh and Williams' model with x normal, the coefficient of variation of $v_2(r)$ decreases as g increases. The coefficients of variation of $v_2(r)$ with g=n and $v_1(r)$ are essentially equal. Further $v_1(r)$ and $v_2(r)$ with g=n are quite stable compared to $v(\bar{y}/\bar{X})$. They have also considered the general regression model where the regression of y on x does not pass through the origin, and C_x is large. Under this non-ideal condition also the coefficient of variation of $v_2(r)$ decreases with g so that g=n is the optimum choice. The coefficients of variation of $v_2(r)$ (with g=n) and $v_1(r)$ are again essentially equal, but both are considerably larger than the coefficient of variation of $v(\bar{y}/\bar{X})$. Rao and Beegle conclude that caution is needed in the indiscriminate use of ratio extimators.

2. Stabilities of Variance Estimators $v(\bar{y}/\bar{X})$ and $v_1(r)$: Asymptotic Theory Assuming a Bivariate Normal Distribution.

Kokan (1963) used the formula for the relative variance of $v_1(r)$ given by Hansen et al (1953 page 177) to compare the stability of $v_1(r)$ with that of $v(\bar{y}/\bar{X})$. This formula was derived by substituting R for r in $v_1(r)$ as a large sample approximation. We will show that this approach is not correct. The correct approach is to expand r in terms of $\delta\bar{x} = (\bar{x}-\bar{X})/\bar{X}$ and $\delta\bar{y} = (\bar{y}-\bar{Y})/\bar{Y}$ and find the variance of $v_1(r)$ for large samples. Using this approach and utilizing the theory of cumulants and k-statistics (Kendall and Stuart, 1958) to find the variance and covariances of sample moments it can be shown that the relative variance (CV^2) of $v_1(r)$ is

$$CV^2[v_1(r)] = \frac{2}{n} + \frac{4}{n}\,\frac{[C_x^2 - \rho C_y C_x]^2}{[C_y^2 - 2\rho C_y C_x + C_x^2]}$$

$$= \frac{2}{n} + \frac{4[\text{Bias}(r)]^2}{V(r)} \qquad (7)$$

to terms of order n^{-1}, where C_y, C_x are coefficients of variation (CV) of y and x respectively and ρ is the coefficient of correlation between y and x. The relative variance of $v(\bar{y}/\bar{X})$ is

$$CV^2[v(\bar{y}/\bar{X})] = \frac{2}{n} \qquad (8)$$

to terms of order n^{-1}.

From (7) and (8) we have

$$CV[v(\bar{y}/\bar{X})] \leq CV[v_1(r)]. \qquad (9)$$

The equality sign in (9) holds only when the regression of y on x is a straight line through the origin. Thus in large samples, with simple random sampling from a bivariate normal population, the coefficients of variation of the variance estimators $v(\bar{y}/\bar{X})$ and $v_1(r)$ are equal only if the regression of y on x is a straight line through the origin; otherwise CV of $V(\bar{y}/\bar{X})$ is always less than that of $v_1(r)$.

It is interesting to note that the Monte Carlo results for small samples obtained by Rao and Beegle (1966) agree with the asymptotic results obtained here, namely, $v(\bar{y}/\bar{X})$ and $v_1(r)$ are equally stable if the regression of y on x is through the origin; otherwise $v_1(r)$ is always less stable. Further, since $|\text{Bias}\ (r)|/\sigma r \leq C_x/\sqrt{n}$ we have to terms of order 1/n.

$$CV^2[v_1(r)] \leq \frac{2}{n} + \frac{4C_x^2}{n} \qquad (10)$$

Kokan had obtained the expression on r.h.s. of (10) for $CV^2[v_1(r)]$ using the general formula given by Hansen at el (1953), which clearly over estimates the coefficient of variation of $v_1(r)$ so far as the large sample approximation to terms of order n^{-1} is concerned. As a result, he had found CV of $v_1(r)$ to be considerably higher than that of $v(\bar{y}/\bar{X})$ even when the regression is through the origin.

The relative stability of two variance estimators may be judged by the ratio of their relative variances. The relative stability of $v_1(r)$ compared to $v(\bar{y}/\bar{X})$ is given by

$$E_1 = CV^2[v(\bar{y}/\bar{X})]/CV^2[v_1(r)]$$

$$= [1+2C_y^2K^2(K-\rho)^2/(1-2\rho K+K^2)]^{-1} \qquad (11)$$

$$= 1 \text{ if } K = \rho \quad \text{(regression through the origin)}$$

$$< 1 \text{ otherwise} \qquad (12)$$

where $K = C_x/C_y$. The relative stability clearly depends on ρ, C_y and $K = C_x/C_y$. The stability of $v_1(r)$ relative to that of $v(\bar{y}/\bar{X})$ is of interest only when the estimator r is more efficient than $\bar{y}/\bar{X}$ (i.e. when $\rho > K/2$). Consequently, the numerical values of E_1 for selected values of $\rho(>K/2)$ K and C_y are presented in Table 1. It will be seen from Table 1 that for fixed C_y, E_1 decreases as $|\rho-K|$ (i.e. departure from regression through the origin) increases. The stability of $v_1(r)$ is low when $C_x = KC_y$ is large.

3. Stability of Variance Estimators $v(\bar{y}/\bar{X})$ and $v_1(r)$: Exact Theory assuming x has a gamma distribution.

In this section we assume that $y_i=\alpha+\beta x_i+u_i$, where u_i's are independent normals with mean zero and variance $n\delta$ (δ is of order n^{-1}) and the variates x_i/n have the gamma distribution with parameter h so that $\bar{x} = \Sigma x_i/n$ has the gamma distribution with parameter m = nh. Under this model we derive the formulae for the variance estimators and investigate their stabilities. All our results are exact for any sample size, n.

The variance of $\bar{y}/\bar{X}$ under the model is given by

$$V(\bar{y}/\bar{X}) = \frac{\delta}{m^2} + \frac{\beta^2}{m} \qquad (13)$$

and $v(\bar{y}/\bar{X})$ is an unbiased estimator of $V(\bar{y}/\bar{X})$. The variance of r is

$$V(r) = \frac{\alpha^2}{(m-1)^2(m-2)} + \frac{\delta}{(m-1)(m-2)}. \qquad (14)$$

The bias of $v_1(r)$ as an estimator of V(r) can be shown to be

$$\text{Bias}\ [v_1(r)] = -\frac{(5m^2-5m+2)\alpha^2}{m^2(m^2-1)(m-1)(m-2)}$$

$$-\frac{2(m^2+2m-2)\delta}{m^2(m^2-1)(m-2)}$$

$$= c_3\alpha^2 + c_4\delta \quad \text{(say)}. \qquad (15.)$$

We note that for finding the variances of $v(\bar{y}/\bar{X})$ and $v_1(r)$, expected values of some functions of sample moments are needed. The method of evaluating these expectations is same as that of Rao and Webster (1966). The details of

Table 1. Asymptotic relative stability of $v_1(r)$ for selected values of K, C_y and $\rho(>k/2)$.

K	C_y	$\rho = .3$	$\rho = .5$	$\rho = .7$	$\rho = .9$
.5	1.0	98	100	96	81
	2.0	92	100	87	52
	3.0	84	100	75	33
1.0	0.5	-	89	93	98
	1.0	-	67	77	91
	2.0	-	33	45	71
1.5	0.5	-	-	-	58
	1.0	-	-	-	25
	2.0	-	-	-	8

evaluating these expectations, which involve some tedious algebra, are omitted and only the final results are given. The variance of $v(\bar{y}/\bar{X})$ is given by

$$V[v(\bar{y}/\bar{X})] = \frac{2\delta^2}{m^4(n-1)} + \frac{4\beta^2\delta}{m^3(n-1)} + \frac{\beta^4}{m^3}[\theta(m+1)(m+2)(m+3)-m]. \qquad (16)$$

The variance of $v_1(r)$ can be shown to be

$$V[v_1(r)] = \frac{\delta^2}{m^4}[3\,\theta + \frac{(n+1)(m+3)}{(n-1)(m+1)} - \frac{(m+2)^2}{(m+1)^2}] + \frac{2\alpha^2\delta}{m^4}[3\,\theta + \frac{(2m-n+3)}{(n-1)(m+1)^2}] + \frac{\alpha^4}{m^4}[\theta - \frac{1}{(m+1)^2}] \qquad (17)$$

where

$$\theta = \frac{[(n+1)(m+6)-12]}{(n-1)(m+3)(m+2)(m+1)}. \qquad (18)$$

From (15) and (17) the MSE of $v_1(r)$ can be obtained as

$$MSE[v_1(r)] = c_5\,\delta^2 + c_6\alpha^4 + c_7\alpha^2\delta \quad \text{(say)} \qquad (19)$$

where the coefficients c_5, c_6 and c_7 are functions of m and n.

Further, we note that in terms of the model

$$\alpha = \bar{Y}[(K-\rho)/K]$$
$$\beta = \bar{Y}[\rho/(Km)]$$
$$\delta = \bar{Y}^2[(1-\rho^2)/(K^2m)] \qquad (20)$$

where $K = C_x/C_y$. Now, using (13) through (18) and (20) the relative variances (i.e. CV^2) of the variance estimators $v(\bar{y}/\bar{X})$ and $v_1(r)$ can be obtained as functions of $K,\rho,$ m and n. At present, we are evaluating CV^2 of the variance estimators to compare their exact stabilities for different values of K, ρ, m and n.

4. Bias of the Jack-Knife Variance Estimator.

In this section we investigate the bias of $v_2(r)$ and compare with that of $v_1(r)$ under the model of section 3. The Jack-Knife variance estimator $v_2(r)$ can be written as

$$v_2(r) = g^{-1}(g-1)^{-1}\sum_1^g(r_{Qj} - r_Q)^2$$

$$= \frac{(g-1)}{g}\Big[\alpha^2\sum_1^g\{\frac{1}{\bar{x}'_j} - \frac{1}{g}\sum_1^g\frac{1}{\bar{x}'_j}\}^2 + \sum_1^g\{\frac{\bar{u}'_j}{\bar{x}'_j} - \frac{1}{g}\sum_1^g\frac{\bar{u}'_j}{\bar{x}'_j}\}^2 + 2\alpha\sum_1^g\{\frac{1}{\bar{x}'_j} - \frac{1}{g}\sum_1^g\frac{1}{\bar{x}'_j}\}\{\frac{\bar{u}'_j}{\bar{x}'_j} - \frac{1}{g}\sum_1^g\frac{\bar{u}'_j}{\bar{x}'_j}\}\Big] \qquad (21)$$

where $\bar{u}'_j$ and $\bar{x}'_j$ are the sample means obtained after omitting the j th group. Now, since $[(g-1)\bar{x}'_j]/g$ has the gamma distribution with parameter $(n-p)h=[m(g-1)]/g$, we have

$$E[\frac{1}{\bar{x}'^2_j}] = \frac{(g-1)^2}{[m(g-1)-g][m(g-1)-2g]}.$$

For $\bar{u}'_j$'s we have the following expected values:

$$E(\bar{u}'^2_j) = \frac{g}{g-1}\delta$$

and

$$E[\bar{u}'_j\ \bar{u}'_j] = \frac{g(g-2)}{(g-1)^2}\delta \quad ; i \neq j, g \geq 3.$$

Using these expected values it can be shown that the expected value of $v_2(r)$ for $g \geq 3$ (the special case of $g = 2$ is discussed in the next section) is

$$E[v_2(r)] = \alpha^2 \{\frac{(g-1)^4}{g[m(g-1)-g][m(g-1)-2g]} - \frac{(g-1)^2}{g} E(\frac{1}{\bar{x}'_i \bar{x}'_j})\} + \delta\{\frac{(g-1)^3}{[m(g-1)-g][m(g-1)-2g]} - (g-2) E(\frac{1}{\bar{x}'_i \bar{x}'_j})\} . \quad (22)$$

From Rao and Webster (1966) we have, for integer m,

$$\frac{g^2}{(g-1)^2} E[\frac{1}{\bar{x}'_i \bar{x}'_j}] = \Gamma(2a+b-2)\Gamma^{-2}(a)\Gamma^{-1}(b)C(a,b) \quad (23)$$

where

$$C(a,b) = \sum_{k=0}^{a-2} (-1)^k \frac{\Gamma^2(a-k-1)\,\Gamma(b+k)}{\Gamma(2a+b-k-2)} + (-1)^{a-1}[2\sum_{k=1}^{a+b-2}(-1)^{k+1}\frac{1}{(a+b-k-1)^2} + (-1)^{a+b}\frac{\pi^2}{6}] \quad \text{if } a \geq 2 \quad (24)$$

$$= 2\sum_{k=1}^{b-1}(-1)^{k+1}\frac{1}{(b-k)^2} + (-1)^{b+1}\frac{\pi^2}{6} \quad \text{if } a=1,\ b \geq 2 \quad (25)$$

$$= \frac{\pi^2}{6} \quad \text{if } a=1,\ b=1 \quad (26)$$

and $a=m/g$ and $b \leq [m(g-2)]/g$.

Now, the bias of $v_2(r)$ as an estimator of $V(r)$ is

$$\text{Bias}[v_2(r)] = E[v_2(r)] - V(r) = c_1\alpha^2 + c_2\delta \text{ (say)}. \quad (27)$$

Using (22) through (26) the coefficients c_1 and c_2 can be expressed explicitly as functions of g and m. However, since the resulting expression would not be in a closed form, it is difficult to investigate analytically the behavior of the bias of $v_2(r)$ as a function of g for fixed m. Therefore, we have made a numerical investigation and the results are presented in Table 2. We find from Table 2 that the bias of $v_2(r)$ decreases monotonically as g increases for fixed m so that the bias is minimum when $g=n$.

We now compare the bias of $v_2(r)$ with that of the customary variance estimator $v_1(r)$ given by (15). The absolute values of the coefficients c_3 and c_4 in the formula for Bias $[v_1(r)]$ decrease as m (≥ 3) increases. These coefficients have been calculated to compare with those in the formula for the bias of $v_2(r)$ and are presented in Table 2. The bias of $v_1(r)$ should be compared with that of $v_2(r)$ (with $g=n$) since the bias of $v_2(r)$ is minimum when $g=n$. From Table 2 we find that the absolute bias of $v_2(r)$ with $g=n$ is less than that of $v_1(r)$ for $n \geq 6$ whenever $m>8$.

5. Stability of the Jack-Knife Variance Estimator.

In this section we investigate the stability of the Jack-Knife variance estimator $v_2(r)$ under the model of section 3 and compare it with that of $v_1(r)$. The variance of $v_2(r)$ is defined by

$$V[v_2(r)] = E[v_2(r)]^2 - E^2[v_2(r)]. \quad (28)$$

For the case of $g=2$, the means obtained from half-samples are independent and therefore the variance formulas are relatively simple. We have

$$v_2(r) = \frac{1}{4}[\alpha^2(\frac{1}{\bar{x}'_1} - \frac{1}{\bar{x}'_2})^2 + (\frac{\bar{u}'_1}{\bar{x}'_1} - \frac{\bar{u}'_2}{\bar{x}'_2})^2 + 2\alpha(\frac{1}{\bar{x}'_1} - \frac{1}{\bar{x}'_2})(\frac{\bar{u}'_1}{\bar{x}'_1} - \frac{\bar{u}'_2}{\bar{x}'_2})] \quad (29)$$

where $\bar{x}'_1$ and $\bar{x}'_2$ are means of first and second half-samples respectively and they are independent gamma variates each with parameter $m/2$. Therefore, we have

$$E(\frac{1}{\bar{x}'^{\,t}_1}) = E(\frac{1}{\bar{x}'^{\,t}_2}) = \frac{1}{\prod_{i=1}^{t}(m-2i)}; \quad t \geq 1.$$

Consequently the expected value of $v_2(r)$ is

$$E[v_2(r)] = \frac{\alpha^2}{(m-2)^2(m-4)} + \frac{\delta}{(m-2)(m-4)}. \quad (30)$$

The bias of $v_2(r)$ as an estimator of $V(r)$ is

$$\text{Bias}[v_2(r)] = \frac{(4m-7)\alpha^2}{(m-1)^2(m-2)^2(m-4)} + \frac{3\delta}{(m-1)(m-2)(m-4)}. \quad (31)$$

Thus the bias of $v_2(r)$ with $g=2$ decreases as m increases.

Now from (29) we have

$$16E[v_2(r)]^2 = E[\alpha^4(\frac{1}{\bar{x}'_1} - \frac{1}{\bar{x}'_2})^4 + (\frac{\bar{u}'_1}{\bar{x}'_1} - \frac{\bar{u}'_2}{\bar{x}'_2})^4 + 6\alpha^2(\frac{1}{\bar{x}'_1} - \frac{1}{\bar{x}'_2})^2(\frac{\bar{u}'_1}{\bar{x}'_1} - \frac{\bar{u}'_2}{\bar{x}'_2})^2].$$

Table 2. The coefficients c_1, c_2, c_3 and c_4 in Bias $[v_2(r)] = c_1\alpha^2 + c_2\delta$ and Bias $[v_1(r)] = c_3\alpha^2 + c_4\delta$ for selected values of m and g.

m	g	$\lvert c_1\rvert \times 10^6$	$\lvert c_2\rvert \times 10^6$	m	$\lvert c_3\rvert \times 10^6$	$\lvert c_4\rvert \times 10^6$
8	2*	3543	17857			
8	4	1961	9082	8	1665	6448
8	8	1500	6513			
10	2	1061	6944	10	634	2980
10	10	479	2573			
12	2	423	3409	12	292	1612
12	4	273	1931			
12	6	233	1571			
12	12	197	1261			
16	2	108	1190	16	88	626
16	4	74	699			
16	8	58	514			
16	16	52	435			
20	2	39	548	20	35	305
20	10	21	227			
20	20	19	198			
24	2	17	296	24	16	171
24	6	11	147			
24	12	9	119			
24	24	9	106			
32	2	5	115	32	5	69
32	16	3	44			
32	32	2	41			

*Note: formula for Bias $[v_2(r)]$ for g = 2 is given in section 4.

On Simplification, this reduces to

$$E[v_2(r)]^2 = \frac{3\alpha^4}{(m-2)^2(m-4)^2(m-6)(m-8)} + \frac{3(m^2-10m+28)\delta^2}{(m-2)^2(m-4)^2(m-6)(m-8)} + \frac{6\alpha^2\delta}{(m-2)(m-4)^2(m-6)(m-8)}. \quad (32)$$

We can obtain $E^2[v_2(r)]$ from (30). Finally the variance of $v_2(r)$ is obtained as

$$V[v_2(r)] = \alpha^4\left[\frac{3}{(m-2)^2(m-4)^2(m-6)(m-8)} - \frac{1}{(m-2)^4(m-4)^2}\right] + \frac{2(m^2-8m+18)\delta^2}{(m-2)^2(m-4)^2(m-6)(m-8)} + \frac{4(m^2+m-18)\alpha^2\delta}{(m-2)^3(m-4)^2(m-6)(m-8)}. \quad (33)$$

From (31) and (33) we can obtain MSE of $v_2(r)$ as

$$MSE[v_2(r)] = c_8\delta^2 + c_9\alpha^4 + c_{10}\alpha^2\delta \quad \text{(say)} \quad (34)$$

where the coefficients c_8, c_9 and c_{10} are functions of $m(>8)$ only.

We have evaluated the coefficients in (34) for selected values of m and those in MSE of

Table 3. The coefficients $c_5 \dots c_{10}$ in MSE Formulas, $MSE[v_1(r)] = c_5\delta^2 + c_6\alpha^4 + c_7\alpha^2\delta$, $MSE[v_2(r)] = c_8\delta^2 + c_9\alpha^4 + c_{10}\alpha^2\delta$ for selected values of m and n.

m	n	$c_5 x10^6$	$c_6 x10^{10}$	$c_7 x10^{10}$	m	$c_8 x10^6$	$c_9 x10^{10}$	$c_{10} x10^{10}$
10	2	251	16736	510772	10	4184	1571043	25103949
10	10	37	6376	125380				
12	2	116	5420	196618	12	871	181480	3622623
12	4	41	2593	80736				
12	6	26	2028	57560				
12	12	13	1566	38597				
16	2	35	966	44448	16	130	11594	323972
16	4	12	406	17153				
16	8	5	246	9354				
16	16	3	182	6235				
20	2	14	259	14163	20	37	1795	64537
20	10	2	49	2326				
20	20	0.8	36	1547				
24	2	7	88	5586	24	14	444	19336
24	6	13	22	1332				
24	12	0.6	13	752				
24	24	0.3	10	500				
32	2	2	16	1294	32	4	52	3154
32	16	0.1	2	128				
32	32	0.07	1	85				

$v_1(r)$, given by (19), for selected values of m and n. They are presented in Table 3. It will be seen from Table 3 that the MSE of $v_2(r)$ with g=2 is considerably larger than that of $v_1(r)$. We conclude that the Jack-Knife variance estimator $v_2(r)$ with g=2 is not very stable. At the present time the investigation of the stability of $v_2(r)$ for general g is in progress and the results will be reported in a subsequent paper.

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